QUARTERLY OF APPLIED MATHEMATICS
VOLUME LV, NUMBER 4
DECEMBER 1997, PAGES 761-767

ON AN INTEGRODIFFERENTIAL EQUATION ARISING IN A THEORY OF PHASE TRANSITIONS IN SOLIDS

By

ROHAN ABEYARATNE (*Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts*)

AND

JAMES K. KNOWLES (*Division of Engineering and Applied Science, California Institute of Technology, Pasadena, California*)

Abstract. This note is concerned with some properties of an integrodifferential equation arising in a continuum model of solid-solid phase transitions.

Introduction. A continuum model of the macroscopic behavior of one-dimensional thermoelastic tensile bars capable of undergoing mechanically or thermally induced solid-solid phase transitions has been investigated in [1–4]. In particular, the discussion in [3] concerns the dynamical response of a doubly-infinite bar composed of the "trilinear" thermoelastic material put forth in [1], with both inertia and heat conduction taken into account. Specifically, the analysis in [3] is directed at an initial-value problem in which, at time $t = 0$, the bar is at a given uniform temperature and is subject to given piecewise-constant distributions of strain and particle velocity, each with a single discontinuity at the origin. One then inquires about the evolution in time of this discontinuity as determined by the fundamental balance principles of momentum and energy, the condition of kinematic compatibility, and the imbalance of entropy. In addition to the Helmholtz free energy potential governing the bulk material behavior, the model introduced in [1] imports from materials science the notion of a kinetic relation controlling the rate at which the phase transition proceeds.

In the specific problem analyzed in [3], the initial strains are in distinct material phases at the given initial temperature. The bar is assumed to occupy the entire x-axis in the reference configuration and, initially, the phase boundary is located at $x = 0$. It is shown that the evolution problem reduces to the determination of the location $x = s(t)$ of the interface between the two phases (i.e., the *phase boundary*) as a function of time. For the particular kinetic relation assumed in [3], the phase boundary velocity $\dot{s}(t)$ is proportional to the thermodynamic driving force f for the phase transition: $\dot{s} = \omega f$, where the constant $\omega > 0$ is the *mobility* of the phase boundary. This kinetic relation leads to a nonlinear integrodifferential equation for $s(t)$. The purpose of the present

Received February 17, 1995.
1991 *Mathematics Subject Classification*. Primary 73B05, 45J05.

note is to bring this equation to the attention of analysts, and to describe some of its properties.

In the next section, we state the integrodifferential equation for $s(t)$ derived in [3], and—taking for granted the existence of a solution $s(t)$ with the properties assumed in deriving the equation—we describe the small-time and large-time asymptotics of $s(t)$. In the third section, we show that, for a special class of initial data, the integrodifferential equation has a solution for which the phase boundary velocity $\dot{s}$ is constant in time. In the final section, we study the stability of this constant-velocity phase boundary.

The integrodifferential equation. The integrodifferential equation satisfied by $s(t)$ as derived in [3] is

$$\dot{s}(t) = -g + \omega r(\dot{s}(t), h) - \kappa \int_0^t \frac{\dot{s}(\tau) r(\dot{s}(\tau), h)}{(t-\tau)^{1/2}} \exp\left\{-\frac{[s(t)-s(\tau)]^2}{4\nu(t-\tau)}\right\} d\tau, \quad t \geq 0. \tag{1}$$

Here $s(t)$ is the unknown location of the phase boundary at time t, which is to be determined. The constant $\nu > 0$ is the thermal diffusivity of the material, and κ is a material constant defined by

$$\kappa = \frac{\omega \rho \lambda_T}{2k\theta_T}(\nu/\pi)^{1/2}, \tag{2}$$

where $\rho > 0$ is the referential mass density of the material, λ_T (which may have either sign) is the latent heat of transition at the transformation temperature $\theta_T > 0$, and the thermal conductivity $k > 0$ is related to the thermal diffusivity ν by the relation $k = \rho c \nu$, $c > 0$ being the specific heat at constant strain for the material. The constants g and h are given in terms of initial data by

$$g = \omega \rho \lambda_T \frac{\theta_0}{\theta_T}, \qquad h = \frac{a(\gamma_R + \gamma_L) + v_R - v_L}{2a}, \tag{3}$$

where $\theta_0 > 0$ is the initial temperature, γ_L and v_L are the prescribed initial values of strain and particle velocity for $x < 0$, and γ_R and v_R are the initial values of these quantities for $x > 0$. The constant $a = (\mu/\rho)^{1/2} > 0$ is the velocity of sound common to both phases, where $\mu > 0$ is the elastic modulus in both phases. Finally, the function $r(\dot{s}, h)$ of phase boundary velocity $\dot{s}$ and the initial datum h is defined by

$$r(\dot{s}, h) = \mu\gamma_T \left[h - \frac{\gamma_T}{2}\left(\frac{a^2 + a\dot{s} - \dot{s}^2}{a^2 - \dot{s}^2}\right)\right] + \rho\lambda_T, \tag{4}$$

where $\gamma_T > 0$ is the transformation strain for the phase transition.

The functional equation (1) is reminiscent in some respects of equations that arise in problems of Stefan type; e.g., see [5].

In deriving (1), it was assumed that the motion of the phase boundary was subsonic in the sense that $|\dot{s}(t)| < a$ for all t; here we make the slightly stronger assumption that there exist constants p and q such that

$$0 < p \leq |\dot{s}(t)| \leq q < a \quad \text{for } t \geq 0. \tag{5}$$

Thus we seek a solution $s(t) \in \mathbf{C}^1([0, \infty))$ of (1) such that $s(0) = 0$, (5) holds, and $\dot{s}(t)$ has a limit $\dot{s}_\infty \in [p, q]$ as t tends to ∞.

Small-time limit of $s(t)$. It follows immediately from (5) and (4) that the integral in (1) is $O(t^{1/2})$ as t tends to 0, so that the small-time limit $\dot{s}_0$ of $s(t)$ exists and satisfies

$$\dot{s}_0 - \omega r(\dot{s}_0, h) + g = 0. \tag{6}$$

Using (4), one can show that, for each given pair g, h, the left side of (6) increases monotonically from $-\infty$ to $+\infty$ as $\dot{s}_0$ increases from $-a$ to a. Thus (6) determines a unique value of $\dot{s}_0$ in $(-a, a)$ for each given set of initial data.

Large-time limit of $s(t)$. Using results established in the Appendix, one can show that the large-time limit $\dot{s}_\infty$ of $\dot{s}(t)$ must satisfy

$$\dot{s}_\infty - (1 - 2\kappa(\pi\nu)^{1/2}/\omega)\omega r(\dot{s}_\infty, h) + g = 0. \tag{7}$$

For simplicity, we now assume that the material constants are such that the coefficient of $r(\dot{s}, h)$ in (7) is positive. Then the left side of (7) increases monotonically from $-\infty$ to $+\infty$ as $\dot{s}_\infty$ increases from $-a$ to $+a$, so that (7) uniquely determines $\dot{s}_\infty$ in $(-a, a)$.

If in the setting of [3] one studies situations in which the two material phases are separated by a phase boundary propagating steadily with constant velocity, one finds that this steady-state velocity must satisfy (7).

Constant-velocity solutions of the integrodifferential equation. Suppose that $s(t)$ is a solution of (1) for which the phase boundary velocity $\dot{s}(t) \equiv \dot{s}_*$ is constant. Then (1) becomes

$$\dot{s}_* = -g + \omega r(\dot{s}_*, h) - \kappa \dot{s}_* r(\dot{s}_*, h) \int_0^t K(\tau)\, d\tau, \tag{8}$$

where

$$K(t) = t^{-1/2} \exp\left(-\frac{\dot{s}_*}{4\nu} t\right). \tag{9}$$

Since $\dot{s}_*$ is to be constant, the coefficient of the integral in (8) must vanish. This implies that either (i) $\kappa = 0$, (ii) $\dot{s}_* = 0$, or (iii) $r(\dot{s}_*, h) = 0$. Since all constants entering the formula (5) for κ are necessarily positive *except* for λ_T, the first possibility can occur only if the latent heat λ_T vanishes. This case, which was studied in detail in [3], is of physical interest when the two "phases" under study are actually variants of a *single* phase, as in the case of martensitic twins. In any event, $\lambda_T = 0$ trivializes the integrodifferential equation (1), and we shall exclude it here. The second case ($\dot{s}_* = 0$) corresponds to an *equilibrium* mixture of phases, which we also exclude at present. Thus only the third possibility remains; from (8), we are then left with the two conditions

$$\dot{s}_* = -g, \tag{10}$$

$$r(\dot{s}_*, h) = 0. \tag{11}$$

When the value of $\dot{s}_*$ given by (10) is inserted in (11), the latter becomes a relation between h and g and thus represents a restriction on the initial data that is necessary for the existence of a constant-velocity solution of (1). By (4), $r(\dot{s}, h)$ increases monotonically from $-\infty$ to $+\infty$ as $\dot{s}$ increases from $-a$ to a for each h, and so the restriction $r(-g, h) = 0$ uniquely determines a value of g such that $|g| < a$ for each h in terms of material parameters. Conversely, if (10) and (11) hold and $|g| < a$, then $s(t) = \dot{s}_* t$ is a constant-velocity solution of (1).

It may be noted that (10) and (11) follow formally from (6) and (7) if it is required that the small-time and large-time limits $\dot{s}_0$ and $\dot{s}_\infty$ coincide.

Stability of constant-velocity solutions. We now investigate the linearized stability of constant-velocity solutions of (1). We assume that the initial data g and h satisfy the condition obtained by eliminating $\dot{s}_*$ between (10) and (11), and we then perturb the initial state by replacing h and θ_0 by $h+\eta$, $\theta_0+\epsilon$, respectively, where η and ϵ are small in a suitable sense. We further assume that the corresponding solution $s(t)$ of (1) is given by $s(t) = \dot{s}_* t + \sigma(t)$, where $\sigma(t)$ is also small. Linearizing (1) about the constant-velocity solution and making use of (10), (11) shows that $\dot{\sigma}(t)$ must satisfy the Volterra integral equation

$$\alpha\dot{\sigma}(t) = \beta + \gamma \int_0^t K(\tau)\, d\tau + \delta \int_0^t K(t-\tau)\dot{\sigma}(\tau)\, d\tau, \tag{12}$$

where $K(t)$ is given by (9), and

$$\alpha = 1 - \omega \frac{\partial r}{\partial \dot{s}}\bigg|_{\dot{s}_*, h} = 1 + \frac{\mu\gamma_T^2 a\omega}{2} \frac{a^2+\dot{s}_*^2}{(a^2-\dot{s}_*^2)^2} > 0, \tag{13}$$

$$\beta = -\omega\rho\lambda_T \frac{\epsilon}{\theta_T} + \omega\eta \frac{\partial r}{\partial h}\bigg|_{\dot{s}_*, h} = -\omega\rho\lambda_T \frac{\epsilon}{\theta_T} + \omega\mu\gamma_T\eta, \tag{14}$$

$$\gamma = -\kappa\dot{s}_*\eta \frac{\partial r}{\partial h}\bigg|_{\dot{s}_*, h} = -\kappa\mu\gamma_T\eta\dot{s}_*, \tag{15}$$

$$\delta = -\kappa\dot{s}_* \frac{\partial r}{\partial \dot{s}}\bigg|_{\dot{s}_*, h} = \frac{\kappa\mu a\gamma_T^2}{2} \frac{a^2+\dot{s}_*^2}{(a^2-\dot{s}_*^2)^2}\dot{s}_* \neq 0. \tag{16}$$

We are interested in whether the solution $\dot{\sigma}(t)$ of (12) remains bounded as t tends to ∞.

If $\beta/\alpha = -\gamma/\delta$, one can easily show that $\dot{\sigma}(t) = -\gamma/\delta =$ constant. If $\beta/\alpha \neq -\gamma/\delta$, one can define a new function φ by

$$\dot{\sigma}(t) = \left(\frac{\beta}{\alpha} + \frac{\gamma}{\delta}\right)\varphi(t) - \frac{\gamma}{\delta}, \tag{17}$$

thereby reducing (12) to a simpler integral equation for φ:

$$\varphi(t) = 1 + \frac{\delta}{\alpha}\int_0^t K(t-\tau)\varphi(\tau)\, d\tau. \tag{18}$$

We note from (13)–(16) that the parameters β and γ depend on the perturbations η and ϵ of the initial data, but δ and α do not. Thus by (18), $\varphi(t)$ does not depend on these perturbations. Clearly the asymptotic behavior for large time of $\dot{\sigma}(t)$ is determined by that of $\varphi(t)$.

The solution of (18) may be found with the help of Laplace transforms; it is given by

$$\varphi(t) = 1 + m\int_0^t K(\tau)\,d\tau + \pi m^2 \int_0^t \exp\{(\pi m^2 - \dot{s}_*^2/(4\nu))\tau\}\operatorname{erfc}(-m(\pi\tau)^{1/2})\,d\tau, \quad (19)$$

where $K(t)$ is given by (9), erfc stands for the complementary error function, and we have written

$$m = \frac{\delta}{\alpha}. \quad (20)$$

We observe from (13), (16), (2), and (20) that the sign of m is that of $\lambda_T \dot{s}_*$, i.e., the product of the latent heat and the phase boundary velocity. By (11) and $(3)_1$, λ_T and $\dot{s}_*$ have opposite signs, and so $m < 0$.

Because of (9), the first integral on the right side of (19) is bounded for large t. Since $m < 0$ at present, one has $\operatorname{erfc}(-m(\pi\tau)^{1/2}) = O(\tau^{-1/2}\exp(-m^2\pi\tau))$ as τ tends to infinity, and the integrand in the second integral in (19) is exponentially small for large τ. It follows that $\varphi(t)$ is bounded for large t. We conclude from (17) that $\dot{\sigma}(t)$ remains small for all time, and hence that the constant-velocity solution $s(t) = \dot{s}_* t$ of (1) is stable. (If $m > 0$, then $\operatorname{erfc}(-m(\pi\tau)^{1/2})$ tends to two as τ tends to infinity, and the second integral in (19) is bounded for large t only if $m < (4\pi\nu)^{-1/2}|\dot{s}_*|$; thus the condition for unboundedness of solutions φ of (18) is $m > 2(\pi\nu)^{-1/2}|\dot{s}_*|$).

Fried [6] has studied the linear stability of a phase boundary moving at constant velocity in a thermoelastic solid capable of existing in two distinct phases. The kinematic setting in [6] is that of anti-plane shear, and so involves *two* space dimensions, rather than one, as in [3] and in the present paper. However, by restricting the perturbations admitted in the stability analysis in [6] to those that do not depend on the coordinate *parallel* to the interface, one can adapt Fried's results to the present context. When this is done, one finds that the results in [6] also predict that the constant-velocity solution of (1) described above corresponds to a stably propagating phase boundary.

Acknowledgment. This work has been partially supported by the U.S. Office of Naval Research through Grant N00014-93-1-0240 to the California Institute of Technology.

Appendix. Here we derive the long-time result (7). Let $s(t) \in C^2([0,\infty))$ be a solution of (1) that satisfies (5) and for which the limiting phase boundary velocity $\dot{s}(\infty) \equiv \dot{s}_\infty$ exists. Set

$$\varphi(t) = \dot{s}(t) r(\dot{s}(t), h). \quad (21)$$

Observe that $\varphi \in C^1([0,\infty))$, φ is bounded on $[0,\infty)$, and the limit $\varphi(\infty) = r(\dot{s}_\infty, h) \equiv \varphi_\infty$ exists. Refer to (1), and let

$$I(t) \equiv \int_0^t \frac{\varphi(\tau)}{(t-\tau)^{1/2}} \exp\left\{-\frac{[s(t)-s(\tau)]^2}{4\nu(t-\tau)}\right\} d\tau. \quad (22)$$

To establish (7), it is only necessary to determine the long-time limit of $I(t)$. Let $T > 0$ be a fixed number to be chosen later, and write

$$I(t) = I_1(t) + I_2(t), \quad (23)$$

where

$$I_1(t) = \int_0^T \frac{\varphi(\tau)}{(t-\tau)^{1/2}} \exp\left\{-\frac{[s(t)-s(\tau)]^2}{4\nu(t-\tau)}\right\} d\tau, \quad t \geq T, \tag{24}$$

$$I_2(t) = \int_T^t \frac{\varphi(\tau)}{(t-\tau)^{1/2}} \exp\left\{-\frac{[s(t)-s(\tau)]^2}{4\nu(t-\tau)}\right\} d\tau, \quad t \geq T. \tag{25}$$

By the mean value theorem, $s(t) - s(\tau) = \dot{s}(\bar{\tau})(t-\tau)$ for some $\bar{\tau}$ between τ and t; by (5), one has $|\dot{s}(\bar{\tau})| \geq p$, so that

$$|I_1(t)| \leq M \int_{t-T}^t \tau^{-1/2} \exp\left\{\frac{-p^2}{4\nu}\tau\right\} d\tau, \quad t > T, \tag{26}$$

where $M = \max_{t\geq 0} |\varphi(t)|$. Thus $I_1(t)$ tends to zero as t tends to ∞. Next, one writes

$$I_2 = I_3 + \varphi_\infty I_4, \tag{27}$$

where

$$I_3(t) = \int_T^t \frac{\varphi(\tau) - \varphi_\infty}{(t-\tau)^{1/2}} \exp\left\{-\frac{[s(t)-s(\tau)]^2}{4\nu(t-\tau)}\right\} d\tau, \quad t \geq T, \tag{28}$$

$$I_4(t) = \int_T^t (t-\tau)^{-1/2} \exp\left\{-\frac{[s(t)-s(\tau)]^2}{4\nu(t-\tau)}\right\} d\tau, \quad t \geq T. \tag{29}$$

From (28), one readily shows that

$$\begin{aligned} |I_3(t)| &\leq \max_{t\geq T} |\varphi(t) - \varphi_\infty| \int_T^t (t-\tau)^{-1/2} \exp\left\{-\frac{p^2}{4\nu}(t-\tau)\right\} d\tau \\ &\leq \frac{2(\pi\nu)^{1/2}}{p} \max_{t\geq T} |\varphi(t) - \varphi_\infty|, \end{aligned} \tag{30}$$

so that $I_3(t)$ can be made arbitrarily small for all $t \geq T$ by choosing T sufficiently large. It then follows from (23) and (27) that

$$\lim_{t\to\infty} I(t) = \varphi_\infty \lim_{t\to\infty} I_4(t). \tag{31}$$

Let

$$u(t,\tau) = \frac{[s(t)-s(\tau)]^2}{4\nu(t-\tau)}, \quad v(t,\tau) = \frac{\dot{s}_\infty^2(t-\tau)}{4\nu}, \quad 0 \leq \tau \leq t, \tag{32}$$

and note that $u \geq p^2(t-\tau)/(4\nu)$, $v \geq p^2(t-\tau)/(4\nu)$. We may write

$$I_4 = I_5 + I_6, \tag{33}$$

with

$$I_5(t) = \int_T^t (t-\tau)^{-1/2}\{\exp[-u(t,\tau)] - \exp[-v(t,\tau)]\}\, d\tau, \quad t \geq T, \tag{34}$$

$$I_6(t) = \int_T^t (t-\tau)^{-1/2} \exp\left[-\frac{\dot{s}_\infty^2}{4\nu}(t-\tau)\right] d\tau, \quad t \geq T. \tag{35}$$

By the mean-value theorem, there is a number w between u and v such that

$$\exp(-u) - \exp(-v) = -[\exp(-w)](u - v); \tag{36}$$

clearly $w \geq p^2/[4\nu(t-\tau)]$; so by (36), (5),

$$|\exp(-u) - \exp(-v)| \leq |u - v| \exp\{-p^2/[4\nu(t-\tau)]\}. \tag{37}$$

Next, one shows using (32) that

$$|u(t,\tau) - v(t,\tau)| \leq \left\{ \frac{a + |\dot{s}_\infty|}{4\nu} \max_{\tau' \geq T} |\dot{s}(\tau') - \dot{s}_\infty| \right\} (t - \tau), \quad T \leq \tau \leq t < \infty, \tag{38}$$

leading from (38), (37), and (34) to the estimate

$$|I_5(t)| \leq \frac{a + |\dot{s}_\infty|}{4\nu} \max_{t \geq T} |\dot{s}(t) - \dot{s}_\infty| \int_T^t (t-\tau)^{1/2} \exp\left[\frac{-p^2}{4\nu(t-\tau)}\right] d\tau. \tag{39}$$

Thus $|I_5(t)|$ can be made arbitrarily small by choosing T sufficiently large. From (31), (33), it then follows that

$$\lim_{t\to\infty} I(t) = \varphi_\infty \lim_{t\to\infty} I_6(t). \tag{40}$$

Finally, it is easy to show that

$$\lim_{t\to\infty} I_6(t) = \int_0^\infty t^{-1/2} \exp(-\dot{s}_\infty^2 t/(4\nu))\, dt = 2(\nu\pi)^{1/2}/\dot{s}_\infty. \tag{41}$$

The original integral $I(t)$ of (22) thus has the property that

$$\lim_{t\to\infty} I(t) = 2(\nu\pi)^{1/2}\varphi_\infty/\dot{s}_\infty = 2(\nu\pi)^{1/2} r(\dot{s}_\infty, h). \tag{42}$$

This establishes (7).

References

[1] R. Abeyaratne and J. K. Knowles, *A continuum model of a thermoelastic solid capable of undergoing phase transitions*, J. Mech. Phys. Solids **41**, 541–571 (1993)

[2] R. Abeyaratne, S.-J. Kim, and J. K. Knowles, *A one-dimensional continuum model for shape-memory alloys*, Internat. J. Solids and Structures **31**, 2229–2249 (1994)

[3] R. Abeyaratne and J. K. Knowles, *Dynamics of propagating phase boundaries: Thermoelastic solids with heat conduction*, Arch. Rational Mech. Anal. **126**, 203–230 (1994)

[4] R. Abeyaratne and J. K. Knowles, *Dynamics of propagating phase boundaries: Adiabatic theory for thermoelastic solids*, Physica D **79**, 269–288 (1994)

[5] J. R. Ockendon, *Techniques of analysis*, in Moving Boundaries in Heat Flow and Diffusion (J. R. Ockendon and W. R. Hodgkins, eds.), Clarendon Press, Oxford, 1975, pp. 138–149

[6] E. Fried, *Stability of a two-phase process involving a planar phase boundary in a thermoelastic solid*, Continuum Mechanics and Thermodynamics **4**, 59–79 (1992)